On the Plane-Width of Graphs

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Abstract

Map vertices of a graph to (not necessarily distinct) points of the plane so that two adjacent vertices are mapped at least a unit distance apart. The plane-width of a graph is the minimum diameter of the image of the vertex set over all such mappings. We establish a relation between the plane-width of a graph and its chromatic number, and connect it to other well-known areas, including the circular chromatic number and the problem of packing unit discs in the plane.

Keywords: plane-width, realization of a graph, chromatic number, circular chromatic number
1 Introduction

This is an extended abstract of [6]. Given a simple, undirected, finite graph \( G = (V,E) \), a realization of \( G \) is a function \( r \) assigning to each vertex a point in the plane such that for each \( \{u,v\} \in E \), \( d(r(u),r(v)) \geq 1 \), where \( d \) is the Euclidean distance. The width of a realization is the maximum distance between the images of any two vertices. In this paper, we introduce a new graph invariant, called the plane-width and denoted by \( pw(G) \), which is the minimum width over all realizations of \( G \). (To avoid trivialities we only consider graphs with at least one edge.)

**Complete graphs.** The problem of determining the plane-width of complete graphs \( K_n \) has previously appeared in the literature in different contexts: finding the minimum diameter of a set of \( n \) points in the plane such that each pair of points is at distance at least one [2], or packing non-overlapping unit discs in the plane so as to minimize the maximum distance between any two disc centers [7]. The exact values of \( pw(K_n) \) are known only for complete graphs on at most 8 vertices. However, the asymptotic behaviour of \( pw(K_n) \) has been determined.

**Theorem 1.1 ([1,2,5])**

\[
\lim_{n \to \infty} \frac{pw(K_n)}{\sqrt{n}} = \left( \frac{2\pi^{-1/2}3^{1/2}}{1} \right)^{1/2} \approx 1.05 .
\]

The plane-width of \( K_4 \) is \( \sqrt{2} \) and our first result is a generalization of this fact.

**Proposition 1.2** The plane-width of every odd wheel is equal to \( \sqrt{2} \).

2 Plane-width and chromatic number

**Small chromatic number.** For small values of the chromatic number, there is a strong relation between the plane-width of a graph and its chromatic number.

**Theorem 2.1** For all graphs \( G \),

(a) \( pw(G) = 1 \) if and only if \( \chi(G) \leq 3 \),

(b) \( pw(G) \notin (1,2/\sqrt{3}] \),

(c) \( pw(G) \in (2/\sqrt{3},\sqrt{2}] \) if and only if \( \chi(G) = 4 \),

(d) \( pw(G) \in (\sqrt{2},2] \) if and only if \( \chi(G) \in \{5,6,7\} \).
In particular, every bipartite graph has plane-width exactly 1. Also, every graph with maximum degree at most 3, different from the complete graph on 4 vertices, has plane-width exactly 1. (By Brooks’s Theorem such graphs are 3-colorable.) The plane-width of every planar graph is at most $\sqrt{2}$ (as such graphs are 4-colorable), and the plane-width of graphs embeddable on a torus is at most 2 (as such graphs are 7-colorable).

**Large chromatic number.** We have already seen in Theorem 1.1 that $pw(K_n) = \Theta(\sqrt{n})$. We show, more generally, that the relation $pw(G) = \Theta(\sqrt{\chi(G)})$ holds for arbitrary graphs as $\chi(G) \to \infty$.

**Lemma 2.2** For every $\epsilon > 0$ there exists an integer $k$ such that for all graphs $G$ of chromatic number at least $k$, it holds that $\chi(G) < \left(\left(\frac{2}{\sqrt{3}} + \epsilon\right) \cdot pw(G)\right)^2$.

**Lemma 2.3** For all graphs $G$, $pw(G) \leq pw(K_{\chi(G)})$.

The two lemmas give a lower and an upper bound which are combined in the following theorem.

**Theorem 2.4** For every $\epsilon > 0$ there exists an integer $k$ such that for all graphs $G$ of chromatic number at least $k$,

$$\left(\frac{\sqrt{3}}{2} - \epsilon\right) \sqrt{\chi(G)} < pw(G) < \left(\sqrt{\frac{2\sqrt{3}}{\pi}} + \epsilon\right) \sqrt{\chi(G)}.$$

Some questions regarding the plane-width of a graph can be answered via the chromatic number by applying Theorem 2.4. For instance, the plane-width of almost every random graph (in the $G_{n,p}$ model with a fixed $p \in (0, 1)$) is $\Theta(\sqrt{n/\log(n)})$ (since the chromatic number of almost every random graph is $\Theta(n/\log(n))$ [3]). Another example is the existence of graphs of arbitrarily large plane-width and girth (as there are graphs of arbitrarily large chromatic number and girth [4]).
Open problem. Let \( \mathbb{P} = \{pw(G) : G \text{ is a graph}\} \). Determine whether there exists a function (a non-decreasing function) \( f : \mathbb{P} \to \mathbb{Z} \) such that \( f(pw(G)) = \chi(G) \) for every non-bipartite graph \( G \).

3 Plane-width and circular chromatic number

The circular chromatic number \( \chi_c(G) \) is a well-known graph invariant and can be seen as a refinement of the chromatic number. We establish a connection between the circular chromatic number and the plane-width.

Lemma 3.1 For all graphs \( G \), \( pw(G) \leq \left[ \sin \left( \frac{\pi}{\chi_c(G)} \right) \right]^{-1} \).

This allows us to apply some known results on the circular chromatic number to prove the existence of graphs with certain plane-widths. Specifically, we obtain the following theorem, which should be viewed as complementing Theorem 2.1.

Theorem 3.2 For every \( \epsilon > 0 \) there exists
(a) A 4-chromatic graph \( G \) such that \( pw(G) < 2/\sqrt{3} + \epsilon \),
(b) A 5-chromatic graph \( G \) such that \( pw(G) < \sqrt{2} + \epsilon \),
(c) An 8-chromatic graph \( G \) such that \( pw(G) < 2 + \epsilon \).

4 Plane-width and graph operations

Homomorphisms and perfect graphs. Any graph with chromatic number \( \chi(G) \) is homomorphic to \( K_{\chi(G)} \). The following lemma generalizes Lemma 2.3.

Lemma 4.1 Let \( G \) be a graph homomorphic to a graph \( H \). Then, \( pw(G) \leq pw(H) \).

We denote by \( \omega(G) \) the maximum size of a clique in \( G \).

Corollary 4.2
(a) For every graph \( G \) and its subgraph \( G' \), \( pw(G') \leq pw(G) \).
(b) For every graph \( G \), \( pw(G) \geq pw(K_{\omega(G)}) \).

These observations together with Lemma 2.3 imply that for graphs whose chromatic number coincides with their maximum clique size, their plane-width is a function of their chromatic number.

Corollary 4.3 Let \( G \) be a graph such that \( \chi(G) = \omega(G) \). Then, \( pw(G) = pw(K_{\chi(G)}) \). In particular, if \( G \) is perfect, then \( pw(G) = pw(K_{\chi(G)}) \).
Cartesian product. Let $G \square H$ be the Cartesian product of $G$ and $H$. Corollary 4.2 implies that $\text{pw}(G \square H) \geq \max\{\text{pw}(G), \text{pw}(H)\}$. In the following theorem, we provide an exact and an asymptotic upper bound on $\text{pw}(G \square H)$.

**Theorem 4.4**

(a) For every two graphs $G$ and $H$,

$$\text{pw}(G \square H) \leq \text{pw}(G) + \text{pw}(H).$$

(b) For every $\epsilon > 0$ there exists a $p > 0$ such that for every two graphs $G$ and $H$ of plane-width at least $p$,

$$\text{pw}(G \square H) \leq \left(\sqrt{\frac{8}{\sqrt{3\pi}}} + \epsilon\right) \max\{\text{pw}(G), \text{pw}(H)\}.$$

Disjoint union. Let $G \uplus H$ denote the disjoint union of $G$ and $H$. By Corollary 4.2, we have $\text{pw}(G \uplus H) \geq \max\{\text{pw}(G), \text{pw}(H)\}$.

**Theorem 4.5** For every two graphs $G$ and $H$, we have that

$$\text{pw}(G \uplus H) \leq \max\left(\text{pw}(G), \text{pw}(H), \frac{1}{\sqrt{3}}(\text{pw}(G) + \text{pw}(H))\right).$$

References


